A model for the coexistence of diffusion and accelerator modes in a chaotic area-preserving map

1993 J. Phys. A: Math. Gen. 26 6231

(http://iopscience.iop.org/0305-4470/26/22/023)

J. Phys. A: Math. Gen. 26 (1993) 6231–6249. Printed in the UK

A model for the coexistence of diffusion and accelerator modes in a chaotic area-preserving map

A N Yannacopoulos and G Rowlands
Physics Department, University of Warwick, Coventry CV4 7AL, UK

Received 26 April 1993, in final form 16 August 1993

Abstract. A random walk model for the coexistence of diffusion and accelerator modes for a chaotic two-dimensional area-preserving map is constructed and solved analytically in order to explain the time behaviour of the numerically calculated diffusion coefficient for such maps.

1. Introduction

Consider an area-preserving chaotic map in x, y which can be brought into a doubly periodic form, that is, can be written as a map of the unit torus $T = [0, 2\pi] \times [0, 2\pi]$. For such maps there exist parts of phase space called accelerator modes [1] where ordered motion occurs rather than stochastic motion. This ordered motion corresponds to constant acceleration of particles to remote parts of phase space and this leads to anomalous enhancement of the diffusion coefficient as calculated for such maps [2, 3]. Examples of maps where such behaviour occurs are the well known standard map [2] or the web map [4, 5].

Our aim is to investigate the effect of the existence of such accelerator modes on the transport through phase space for an area-preserving map (two-dimensional symplectic map). The motivation is to explain the fluctuations observed in the numerically calculated diffusion coefficient $\langle p^2 \rangle / 2n$ where p denotes displacement and n the number of iterates. Usually the asymptotic value of D for large n is constant. However, for many maps the ratio shows oscillatory behaviour and/or variation proportional to n^α for large n. A typical example of the variation of D with n, obtained by numerical iteration of the web map is shown in figure 1. Of course the times of interest are longer than the time needed for the effect of initial conditions to be damped away.

2. Formulation of the problem

The variation of the diffusion coefficient D with n can have very complicated behaviour. We associate this behaviour with the presence of accelerator modes and with regions of non-chaotic behaviour in the phase space. However, the exact structure of the phase space is extremely complicated and some simplification is necessary.

The phase space is modelled as follows. It is assumed to be infinite and two-dimensional. In the space there exists a periodic array of points which corresponds to accelerator modes and which forms an infinite orthogonal lattice of points. Whenever a particle reaches such a point it can make a finite jump to another point of the lattice (that is, to another accelerator mode) rather than diffuse to neighbouring points in the space. We also consider the effects

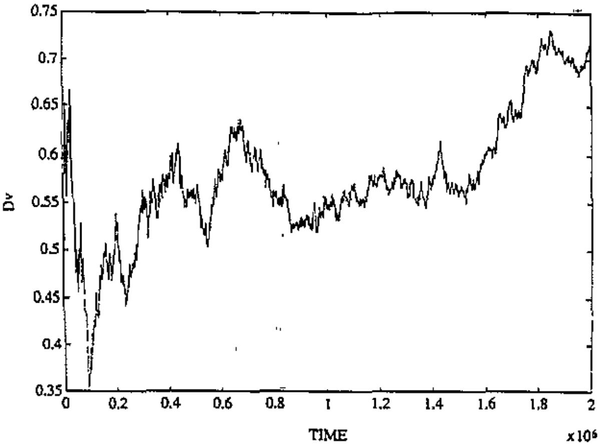

Figure 1. A typical numerical calculation of the diffusion coefficient as a function of time for a map containing accelerator modes.

of the existence of islands surrounding stable periodic points that act as traps in the diffusion process through phase space. The stable periodic points also form an infinite orthogonal lattice in phase space. (The infinity of the lattices of both the accelerator modes and the traps arises because of the double periodicity of the original map.) On every other part of phase space, the motion is approximated by a diffusion process with a constant diffusion coefficient D.

Thus particles diffuse through phase space until they reach the vicinity of an accelerator mode or a stable island (trap). There they can be trapped and start performing finite jumps to other points of the lattice or remain trapped, according to whether the lattice point is an accelerator mode or a stable island. Trapping at lattice points occurs for a finite number of iterations of the map m, with a probability distribution $\bar{\psi}(m)$. Then detrapping occurs and the particles are allowed to diffuse again until they are brought by diffusion to the vicinity of another accelerator mode or island.

Half the accelerator modes correspond to orbits for which $p \to \infty$ as $n \to \infty$ and the other half correspond to orbits for which $p \to -\infty$ as $n \to \infty$. To distinguish between these two types of accelerator modes we will call the second type retarder modes. The coexistence of these two types of modes is found, for example, in the web map.

The accelerator modes exist at the points (kx_A, jy_A), the retarder modes exist at the points (kx_R, jy_R) and the stable islands (traps) exist at the points (kx_T, jy_T) of the phase space where k and j are integers. For simplicity we allow jumps between the accelerator modes to be only in one direction, say the x direction. Generalization of the model to allow for jumps in all directions is straightforward.

2.1. A discrete model

The random walk situation outlined above is a discrete time–discrete space random walk model. The usual random walk model is assumed on a lattice of points one unit distance apart, so that the probability of motion to the left or to the right is equal to $\frac{1}{2}$. A second lattice is embedded on this lattice; this is a lattice of accelerator modes (retarder modes or traps). When a particle first reaches such a point it is reinjected in the normal lattice with probability $(1-\alpha)$ or stays trapped there performing correlated jumps with probability α. The number m of correlated jumps performed by a particle in such a mode is distributed with a probability distribution $\bar{\psi}(m)$. In what follows we assume that the second lattice spacings (x_A for example) are large compared with unity.

On the normal lattice the usual random walk equation for $p(n, t)$ which is the probability that a particle is at lattice site n at time t, where $n, t \in \boldsymbol{Z}$, can be written as

$$p(n,t) = \hat{T}_1(p(n,t)) = \tfrac{1}{2}p(n-1,t-1) + \tfrac{1}{2}p(n+1,t-1). \tag{1}$$

This equation simply states that a point of the normal lattice can be reached only from its nearest neighbours and that it takes a time unit for a particle at $n+1$ or $n-1$ to hop to n.

2.2. *Accelerator modes*

Equation (1) is not valid for accelerator modes and their nearest neighbours. An accelerator mode can be reached not only from its nearest neighbours but also from particles which were in other accelerator modes.

Any particle that just got into the accelerator mode at $(N-s)l$ ($l \equiv x_A$) by diffusion, and is going to spend more than s iterations in this mode, is going to end up in s time units at Nl. This is a process that takes s time units to be completed, so the rate of particles into the accelerator mode at Nl, at time t, due to contributions from other accelerator modes is

$$\alpha \frac{\psi(s)}{s}\frac{1}{2}(p(Nl-sl-1,t-s-1) + p(Nl-sl+1,t-s-1)) \tag{2}$$

where $\alpha\psi(s)$ is the probability that a particle stays in an accelerator mode for more than s iterations.

At any time t, only t accelerator modes at most can contribute to Nl because particles in accelerator modes $(N-s)l$ with $s > t$ have not had sufficient time to reach Nl. However, at time t the particles which just entered $(N-s)l$ at time $t-s$ for $s < t$ and are going to be trapped there for m iterates (where $m > t$) can contribute to Nl at t. Thus the total rate of particles reaching the point Nl at t from other accelarator modes is

$$\sum_{s=1}^{t} \frac{\psi(s)}{s} \hat{T}_1 p(Nl-sl, t-s). \tag{3}$$

The rate out of an accelerator mode is equal to $p(Nl, t-1)$, since everything that was in the accelerator mode will have to leave in one iteration (either to go to some other accelerator mode or back to the diffusion lattice). Thus the probability that a particle is found at Nl at time t is given by the equation

$$p(Nl,t) = \hat{T}_1 p(Nl,t) + a\sum_{s=1}^{t} \Psi(s)\hat{T}_1 p(Nl-sl,t-s). \tag{4}$$

where $\Psi(s) = \psi(s)/s$.

We now consider particles that reach the nearest neighbours of the accelerator modes. First of all it is important to realize that not all the particles which were at Nl at time $t-1$ can contribute to $Nl+1$ and $Nl-1$ at time t. Only those which have finished their sojourn in the accelerator mode lattice are allowed to get back to the normal lattice. The rate of particles into the normal lattice from the site Nl at time t is thus given by

$$p(Nl,t-1) - a\sum_{s=1}^{t} \Psi(s)\hat{T}_1 p(Nl-(s-1)l,t-s). \tag{5}$$

Half of the particles described by equation (5) will go to site $Nl+1$ and the other half to site $Nl-1$. Apart from this, these sites can be reached by normal random walk from sites

$Nl \pm 2$. Thus the probability of being at sites $Nl \pm 1$ at time t is given by

$$p(Nl \pm 1, t) = \frac{1}{2}p(Nl \pm 2, t-1) + \frac{1}{2}p(Nl, t-1) \\ -\alpha\frac{1}{2}\sum_{s=1}^{t}\Psi(s)\hat{T}_1 p(Nl-(s-1)l, t-s). \tag{6}$$

The random walk including the effect of accelerator modes is described by the usual random walk equations plus an effective source term localized on the lattice of accelerator modes and their nearest neighbours, that is

$$p(n,t) = \frac{1}{2}p(n+1,t-1) + \frac{1}{2}p(n-1,t) + S_A \tag{7}$$

where

$$S_A = \alpha\sum_N \delta(n-Nl)\sum_{s=1}^{t}\Psi(s)\hat{T}_1 p(n-sl, t-s) \\ -\tfrac{1}{2}\alpha\sum_N \delta(n-Nl-1)\sum_{s=1}^{t}\Psi(s)\hat{T}_1 p(n-(s-1)l-1, t-s) \\ -\tfrac{1}{2}\alpha\sum_N \delta(n-Nl+1)\sum_{s=1}^{t}\Psi(s)\hat{T}_1 p(n-(s-1)l+1, t-s). \tag{8}$$

It is an easy exercise to show that this model conserves the total number of particles.

2.3. Retarder modes

The source term associated with the retarder modes, S_R, is similar to the one for the accelerator modes, only that it would be concentrated on a different infinite lattice ($l \equiv x_R$) and the terms containing $p(n-sl)$ in S_A will have to be replaced by $p(n+sl)$ in S_R. This is due to the fact that particles in the retarder modes stream in the opposite direction to particles in the accelerator modes.

2.4. Traps

The source term corresponding to the effect of traps on the random walk, S_T, is of a slightly different form. If a particle is caught in a trap, it spends a finite time in the trap before being released back into the normal random walk. A particle that is driven into a trap by the random walk will stay on this site for m time units with a probability $\alpha r(m)$ and then leave the trap to go back to the random walk. Note that $r(s)$ is the first exit probability distribution which is related to the survival probability $\bar{\psi}(s)$ (that is, the probability that a particle starting in a trap at $t = 0$ is still in the trap at $t = s$) by the simple relation $r(s) = -\mathrm{d}\bar{\psi}(s)/\mathrm{d}s$. The probability distribution for particles spending more than time t in the trap is simply the integral of $\bar{\psi}(s)$ [8].

The rate at which particles enter the trap at time t is simply that getting into the trap via diffusion: $\hat{T}_1 p(n_T, t)$. If a fraction α of all the particles that land in a trap are detained there for an infinite amount of time, then the rate out at time t would just be a fraction $(1-\alpha)$ of what was in the trap at time $t-1$, that is, $(1-\alpha)p(n_T, t-1)$. However, we allow for the possibility for particles that were trapped at time $t-m$ to be released from the trap, back to normal diffusion, at some later time t. Such particles will enhance the rate of particles out of the trap at time t. A particle first caught in the trap at time $t-s$ will be released from the trap with probability $\alpha r(s)$, and get back into the normal diffusion. Hence, the total rate out of the trap is

$$(1-\alpha)p(n_T, t-1) + \alpha \sum_{s=1}^{t} r(s)\hat{T}_1 p(n_T, t-s). \tag{9}$$

The rate into the nearest neighbours of the trap site $n_T \pm 1$ is the normal rate corresponding to diffusion from $n_T \pm 2$ plus half the rate out of the trap site. The rate out of the neighbouring sites is $p(n_T \pm 1, t-1)$ since everything on these sites will have to leave in one iteration.

Following the same reasoning as in the case of the accelerator modes, we see that the source term S_T is of the form ($l \equiv x_T$)

$$\begin{aligned} S_T = {} & \alpha \sum_N \delta(n - Nl - n_T)\left(p(n, t-1) - \sum_{s=1}^{t} r(s)\hat{T}_1 p(n, t-s)\right) \\ & -\tfrac{1}{2}\alpha \sum_N \delta(n - Nl - n_T - 1)\left(p(n-1, t-1) - \sum_{s=1}^{t} r(s)\hat{T}_1 p(n-1, t-s)\right) \\ & -\tfrac{1}{2}\alpha \sum_N \delta(n - Nl - n_T + 1)\left(p(n+1, t-1) - \sum_{s=1}^{t} r(s)\hat{T}_1 p(n+1, t-s)\right). \end{aligned} \tag{10}$$

Note that in the above source term, all the probability functions are calculated at the same point because the particle is static for some time after it has been trapped. It is also straightforward to check that the source term S_T conserves probability. This is consistent with the fact that a particle is counted when it is temporarily immobilized in a trap and it is not considered as lost from the system.

2.5. Continuous model

The discrete model proposed in the previous subsection can be written in a continuous form which is more useful for analytical and numerical approximation. We assume that the distance between two ordinary lattice points is infinitesimal compared with the length scales of the probability function, but that the distance between two accelerator modes x_A is kept finite. The time taken by a hop between two normal sites is also taken to be infinitesimal so that time can be treated as a continuous variable. In order to avoid regions of space with infinite particle velocities, the jumps between accelerator modes take a finite but small time. Then in the continuous limit equations (7) and (8) reduce to

$$\frac{\partial p}{\partial t} - D\nabla^2 p = \alpha S_A \tag{11}$$

where

$$\begin{aligned} S_A = {} & \sum_N \delta(x - Nx_A)\delta(y - Ny_A) \\ & \times \sum_{s=1}^{\infty} \Psi(s) H(s-t)(p(x - sx_A, y, t-s) - p(x - (s-1)x_A, y, t-s)) \end{aligned} \tag{12}$$

and where $H(s-t)$ denotes the Heaviside function. In the above derivation we have assumed that the normal random walk, or diffusion, takes place on a two-dimensional lattice but the accelerator modes make particles stream only in the x direction.

This equation is a diffusion equation with a localized source term on the accelerator mode lattice. Note that the discrete model had a source term localized on the accelerator mode lattice and the nearest neighbours but here the source term is replaced by one localized on the accelerator modes only, because these lattice sites are regarded as one.

The continuous form of the source term for the retarder modes, S_R, is obtained in an analoguous fashion.

The continuous analogue of the source term for the traps, S_T, is of a slightly different form, namely

$$S_T = -\frac{\alpha}{2}\sum_{N,L}\nabla^2[\delta(x - Nx_T - x_0)\delta(y - Ly_T - y_0)(p(x, y, t) - \sum_{s=1}^{t} r(s)p(x, y, t - s))]. \tag{13}$$

Summarizing, we see that both in the discrete and the continuous cases, our basic model is to consider that over the entire phase space a diffusion equation with a constant diffusion coefficient is applicable except on the accelerator and retarder modes and the stable islands. The transfer of particles from one accelerator mode to another and the effect of the trapping of particles in the stable islands is modelled by adding effective sources localized on the lattice of structures to the diffusion equation.

3. Solution of the equations

Although equations such as (11) and (12) or their discrete analogues can be solved exactly (see appendix C), the solution is extremely complicated. Below we give an iterative scheme based on the smallness of α, which is a reasonable procedure for the case where most of the phase plane is chaotic. This is particularly useful when combined with the fact that we are only interested in the low moments of the probability function which is all that is necessary for the calculation of an effective diffusion coefficient. The method is given here for the continuous case but it is readily extended to treat the discrete case. The details are given in appendix 2.

We write our equation in the more compact operator form

$$\hat{D}p(x, y, t) = \epsilon\hat{L}p(x, y, t) + \delta(x - x_0)\delta(y - y_0) \tag{14}$$

where $\hat{D}$ is the diffusion operator and we have introduced a real source of particles at the point x_0, y_0. Here $\hat{L}p(x, y, t)$ represents the effective source term and ϵ a small parameter associated with α. For $\epsilon = 0$ the solution of (14) is just Green's function of the normal diffusion equation and is given by [6]

$$p^{(0)}(x, y, t) = \frac{D}{4\pi(t - t_0)}\exp\left(-D\frac{(x - x_0)^2 + (y - y_0)^2}{4(t - t_0)}\right)H(t - t_0) \tag{15}$$

where $H(t - t_0)$ is the Heaviside function. Then by writing $p = p^{(0)} + \epsilon p^{(1)} + O(\epsilon^2)$ we find

$$\hat{D}p^{(1)}(x, y, t) = R(x, y, t) \tag{16}$$

where $R(x, y, t) = \hat{L}p^{(0)}(x, y, t)$ is a known function of x, y, t. The solution to this equation is given by

$$p^{(1)} = \int G(x, y, t|x', y', t')R(x', y', t')\,dx'dy'dt' \tag{17}$$

where $G(x, y, t \mid x', y', t')$ is Green's function for the operator $\hat{D}$ and is given by (15) with x', y', t' replacing x_0, y_0, t_0. The integrations with respect to x' and y' are over the whole space and the integration with respect to t' is from 0 to t. Then, to first order, the correction to the distribution function is given by

$$p^{(1)}(x,y,t) = \int G(x',y',t' \mid x,y,t)(S_{A,R}(p^{(0)}(x',y',t')) + S_T(p^{(0)}(x',y',t'))) \, \mathrm{d}x'\mathrm{d}y'\mathrm{d}t' \tag{18}$$

where we denote by $S_{A,R}$ the sum of the source terms corresponding to the accelerator modes and the retarder modes.

4. Calculation of the diffusion coefficient

The quantities we are primarily interested in are the moments of the probability distribution $p(x,y,t)$. We define two effective diffusion coefficients D_x and D_y by

$$D_x(t) = \frac{M_{2,x}(t)}{tM_0} \qquad \text{and} \qquad D_y(t) = \frac{M_{2,y}(t)}{tM_0} \tag{19}$$

where

$$M_{2,x}(t) = \int x^2 p(x,y,t)\,\mathrm{d}x\mathrm{d}y = \langle x^2 \rangle \tag{20}$$

$$M_{2,y}(t) = \int y^2 p(x,y,t)\,\mathrm{d}x\mathrm{d}y = \langle y^2 \rangle \tag{21}$$

$$M_0(x) = \int p(x,y,t)\,\mathrm{d}x\mathrm{d}y \tag{22}$$

and the integrations are over the entire space. These diffusion coefficients characterize the motion over the whole of phase space which may now be taken to be uniform. Importantly, $D_x(t)$ and $D_y(t)$ are the diffusion coefficients which are to be compared with values of $\frac{\langle p_x^2 \rangle}{n}$ and $\frac{\langle p_y^2 \rangle}{n}$ obtained by iterating the maps in numerical experiments. In particular, we are interested in the behaviour of $D_x(t)$ and $D_y(t)$ as functions of time for our simple stochastic model.

After some cumbersome algebra we can express the moments in the form

$$M_{2,x}(t) = Dt + A_1 + A_2 + A_T \tag{23}$$

$$M_{2,y}(t) = Dt. \tag{24}$$

The functions A_1, A_2 and A_T, which are functions of t, are given explicitly in appendix A. The zeroth moment M_0 is always equal to 1, because of the fact that our model preserves the number of particles. In appendix B this perturbation method is briefly sketched for the discrete model, and is shown to give essentially the same results.

5. Results

The diffusion coefficients for the x and y directions have been calculated using the analytical formulae obtained above (see appendices A and B) for the particular case where the trapping time distribution is of the form

$$r(m) = \sum_i^L A_i \delta(m - M_i). \tag{25}$$

The value of the parameter α is taken to be of the order of 10^{-3}. The calculated value of D_x as a function of t for $L = 1$, for accelerator modes only is shown in figure 2. We note that the diffusion coefficient $D_x(t)$ shows variation with time in the shape of a large bump which corresponds to trapping in the accelerator mode for a finite time. The introduction of L terms in $r(t)$ produces L bumps in D_x. After each bump the diffusion coefficient $D_x(t)$ relaxes slowly to a constant value D_1, larger than D, so that the effect of the particle being trapped in an accelerator mode for a finite number of iterations leads to the enhancement of the effective diffusion coefficient measured at large times. An asymptotic analysis of the model given in appendix D confirms this behaviour.

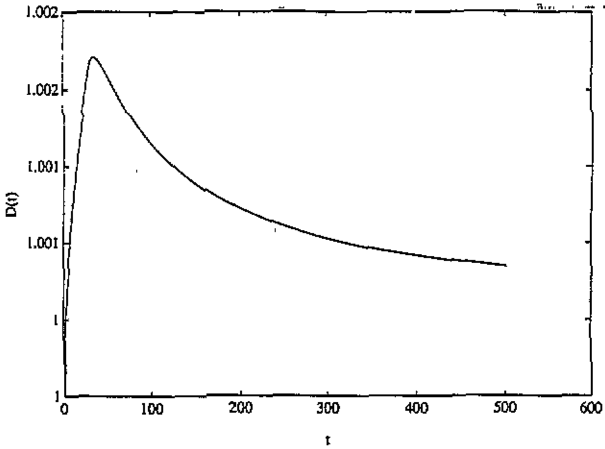

Figure 2. Diffusion coefficient calculated from the results of our model in the case of accelerator modes and a delta-function trapping probability distribution.

If we now consider the model including traps only, we find a dip in the diffusion coefficient; this is illustrated in figure 3.

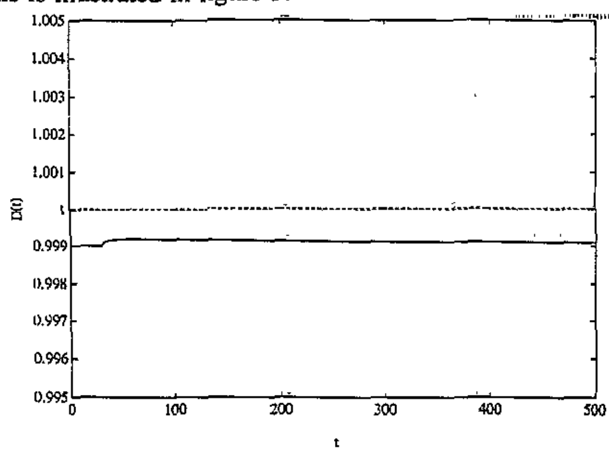

Figure 3. Diffusion coefficient calculated from the results of our model in the case of traps only, with a delta-function trapping probability. The dashed line is the diffusion coefficient in the case of no traps. The bump is due to the release of particles from the trap after a time lag.

The oscillations (fluctuations) observed in $D_x(t)$ are similar to the ones found in the calculated diffusion coefficients obtained from numerical simulations of maps (see figure 1 where the diffusion coefficient is plotted as a function of time for the web map). The multiple trapping in an accelerator mode which is assumed in our model in order to get more than one 'bump' in our theoretical diffusion coefficient is manifested by the large-scale structure of these results. In figure 4 we show a single orbit of the web map which undergoes multiple trapping in an accelerator mode. Therefore, the multiple delta-function-type trapping distribution considered here models, at least qualitatively, the true particle dynamics.

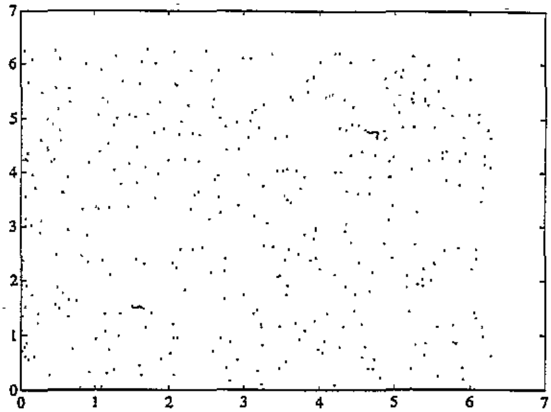

Figure 4. A typical single orbit of the web map. The two small continuous loops show the existence of multiple trapping in the accelerator modes that can be modelled by a multiple delta-function trapping distribution.

As expected, the behaviour of $D_y(t)$ does not show any significant fluctuations since we only allow the accelerator modes to be connected in the x direction. In $D_y(t)$ we just see the effect of traps.

Ishizaki *et al* [7], using a method based on a statistical mechanics formalism of dynamical systems, estimated the long-time behaviour of the diffusion coefficient for the case of the repeated sticking to an accelerator mode. Assuming that the probability for an orbit to stick in such a mode for longer than n iterates is of the power-law form $\psi(n) \sim n^{-(\beta-1)}$ for $n \gg 1$ and $2 > \beta > 1$, they found that the diffusion coefficient for orbits that stick to the accelerator modes is $D \sim n^{2-\beta}$. Taking into account these orbits, as well as orbits that diffuse without getting trapped, the diffusion coefficient is of the form $D(n) = D_1 + D_2 n^{2-\beta}$.

We now apply our method to discuss this case by assuming that $\bar{\psi}(n) \sim n^{-\beta}$ for $n \gg 1$. Then, as discussed in section 2.1, this implies $\psi(n) \sim n^{-(\beta-1)}$ and $r(n) \sim n^{-1-\beta}$.

The major difference between the delta function-like distribution function and this power law is that in the first, detrapping is ensured whereas for a power-law distribution function the possibility of trapping exists for an infinite number of iterations. Using the results of appendices 3 and 4 we find that $D_x(t) = D_1 + D_2 t^{-\beta+2}$ as $t \to \infty$, which is identical to the result obtained by Ishizaki *et al* [7]. The second moment and the diffusion coefficient, as calculated by our method for the case of a power-law trapping distribution taking into account only the accelerator modes, are plotted in figure 5 om the entire time scale. Our results are in good agreement with those of Ishizaki *et al* [7] obtained by direct iteration of the standard map.

In the case where only the trap terms are present, the asymptotic time dependence is of the form $D_x(t) = D_1 + D_2 t^{-\beta}$ which corresponds to a diffusion coefficient decaying, since $1 < \beta < 2$, to a constant value D_1. Note that the constant terms D_1 in the above expressions are not equal to D (the background diffusion constant).

The present model predicts another interesting result concerning the effect of the form of the trapping distribution $\hat{\psi}$ on the asymptotic time behaviour. Namely, it points to a connection between the microscopic properties of the random walk, which is actually an approximation of motion in the connected chaotic regions of the phase space (trapping

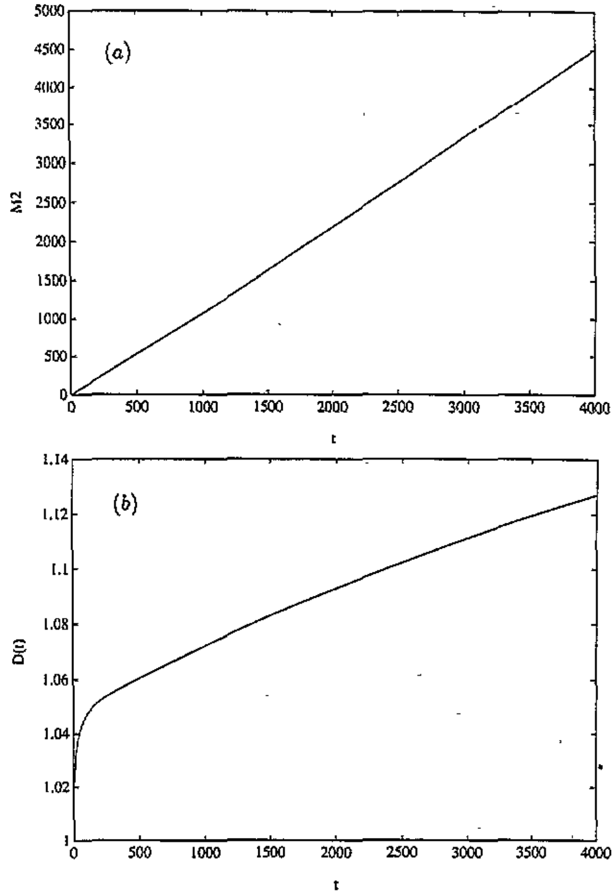

Figure 5. (*a*) Second moment and (*b*) diffusion coefficient for the case of accelerator modes, considering a distribution function for the trapping times with a power-law decay.

distribution in the lattice sites of the accelerator modes), and its macroscopic and easily measurable properties, such as the asymptotic time behaviour of the diffusion coefficient. For an exponentially decaying trapping distribution function $\psi(m) = A\exp(-\lambda m)$ one obtains a diffusion coefficient independent of time. That is, the accelerator modes show no observable effects on the asymptotic time dependence of the effective diffusion coefficient. The details of the calculation are presented in appendix 4. This is of course in contrast to the case of a power-law trapping distribution in the accelerator modes where the diffusion coefficient has a power-law asymptotic behaviour in time and then the effect of the accelerator modes is shown in the asymptotic behaviour of the random walk.

In a recent paper Zaslavskii and Tippet [9] studied the statistical behaviour of a dynamical system with long flights (jets) in certain parts of phase space, and focused their attention on the effect of the Poincaré recurrence statistics on the asymptotic behaviour of diffusion. According to their extensive numerical results, the diffusion coefficient for the dynamical system in question approaches a constant large-time value for certain parameter values for which the Poincaré recurrence statistics follow an exponential law. In contrast, in the case where parameter values were chosen in such a way that the Poincaré recurrence statistics are power-law, the diffusion coefficient diverges asymptotically in time, also following the power law.

We identify the integral of the Poincaré recurrence probability distribution function in the parts of phase space associated with the existence of long flights with the trapping time distribution $\bar{\psi}(m)$ in the accelerator modes used in our model. Hence, the results that Zaslavskii and Tippet [9] obtained by extensive numerical calculations follow immediately from our analysis. Namely, when the Poincaré recurrence statistics follow a power law, $\Psi(m)$ and D also follow power laws. When the Poincaré recurrence statistics follow an exponential law then $\Psi(m)$ also follows an exponential law but D is now constant.

6. Conclusion

We have constructed a simple stochastic model describing the coexistence of accelerator modes, stable islands and diffusion for area-preserving chaotic maps. The analytically predicted forms for the effective diffusion coefficient of this simple model exhibit all the qualitative behaviour obtained by direct numerical iteration of the maps. The different asymptotic time behaviours found in various numerical simulations can be explained in terms of different trapping probability functions $\Psi(m)$.

Our model is shown to be consistent in the asymptotic time limit with the work of Ishizaki *et al* [7]. However, our treatment of the problem using rate equations enables us to obtain results valid for all time whereas the treatment in [7] is purely asymptotic.

Furthermore, the asymptotic results of our model are shown to coincide with the numerical observations of Zaslavskii and Tippet [9] in the case of a chaotic flow in the presence of jets. These results show that there is a link between the microscopic properties (the Poincaré recurrence) and the macroscopic properties (the time dependence of the diffusion coefficient) of the motion.

Finally, even though our model has been formulated for a very simple rectangular lattice of points having a periodic infinite array of structures (accelerator modes or traps), which is the situation that corresponds to an area-preserving map of the torus, the generalization to more general lattices is possible and straightforward. Treating the structures as being on a lattice of points is of course an approximation but the complexity due to finite size regions can reasonably be absorbed into the definition of the $\Psi(m)$'s. Although a generalization to three dimensions is straightforward, the asymptotic results are expected to be different since for random walks of dimensions higher than two, the probability that a diffusion particle reaches any particular point, for example a trapped site, is no longer equal to 1.

Acknowledgments

This work was accomplished while one of the authors (ANY) held a Hellenic State Scholarship Foundation (SSF) grant.

Appendix A.

In this appendix we give explicitly the algebraic forms of the functions involved in the calculation of the diffusion coefficients.

$$A_1 = \int_0^t \mathrm{d}t' \sum_{nl} \sum_{m=1}^{[t']} \Psi(m) \frac{2m-1}{t'-m} \exp\left(-\frac{(n-x_0)^2 + (l-y_0)^2}{t'-m}\right) H(t'-m) \tag{A1}$$

where x_0 and y_0 are the starting points of the particle. If we assume that $(x_0, y_0) \neq (0, 0)$ then the above expression can be simplified to

$$A_1 \simeq \int_0^t \sum_{m=1}^{[t']} \Psi(m) \frac{2m-1}{t'-m} \exp(-\frac{x_0^2+y_0^2}{t'-m}) + \int_0^t \sum_{m=1}^{[t']} \frac{2m-1}{t'-m} \left(\frac{2K}{\pi} - 1\right) \tag{A2}$$

where K is the complete elliptic integral and

$$q = \exp\left(-\frac{1}{t'-m}\right) = \exp\left(-\frac{\pi K'}{K}\right). \tag{A3}$$

The term A_2 for the retarder modes is similar.

Appendix B.

In this appendix the first-order perturbative solution is obtained for the discrete random walk model presented in section 2.1. and is shown to agree with the results obtained from the first-order perturbative solution of the continuous model.

The probability distribution for the discrete model to first order in α is given by

$$p^{(1)} = \sum_{t'=0}^{t} \sum_{n'} G(n-n', t-t')(S_A(p^{(0)}(n', t')) + S_T(p^{(0)}(n', t'))) \tag{B1}$$

where

$$G(n-n', t-t') = \int_{-\pi}^{\pi} \exp(\mathrm{i}k(n-n'))(\cos(k))^{(t-t')}\, \mathrm{d}k \tag{B2}$$

is the Green function for the simple random walk on a lattice and

$$p^{(0)}(n', t') = G(n'-n_0, t'). \tag{B3}$$

The correction to the probability distribution of the normal random walk, due to the accelerator modes, to first order in α is then

$$\begin{aligned} p^{(1)}(n,t) = &\sum_{N,t'}^{t} \int_{-\pi}^{\pi} \mathrm{d}q (\cos q)^{t-t'} \exp(\mathrm{i}(n-Nl)q) \sum_{s=1}^{t'} \Psi(s) \hat{T}_1 p(Nl - sl, t'-s) \\ &- \sum_{N,t'}^{t} \int_{-\pi}^{\pi} \mathrm{d}q (\cos q)^{t-t'} \cos q \exp(\mathrm{i}(n-Nl)q) \\ &\times \sum_{s=1}^{t'} \Psi(s) \hat{T}_1 p(Nl - (s-1)l, t'-s) \end{aligned} \tag{B4}$$

From this equation it is obvious that the Fourier transform of this correction term is

$$\begin{aligned} F(p^{(1)}) = f(q) = &\sum_{N,t'}^{t} (\cos q)^{t-t'} \exp(-\mathrm{i}Nlq) \sum_{s=1}^{t'} \Psi(s) \hat{T}_1 p(Nl - sl, t'-s) \\ &- \sum_{N,t'}^{t} (\cos q)^{t-t'} \cos q \exp(-\mathrm{i}Nlq) \sum_{s=1}^{t'} \Psi(s) \hat{T}_1 p(Nl - (s-1)l, t'-s) \end{aligned} \tag{B5}$$

and since we are interested in first-order in α we will substitute the source terms appearing in this expression by

$$\hat{T}_1 p(x,t) = \int_{-\pi}^{\pi} \mathrm{d}k (\cos k)^t \exp(\mathrm{i}kx). \tag{B6}$$

The correction to the zeroth moment which is the total number of particles due to the existence of the accelelator modes is given by

$$\Delta M_0 = f(0) = 0 \tag{B7}$$

as expected by particle conservation. The correction to the second moment, due to the existence of the source term related to the accelerator modes, is given by

$$\Delta M_2 = -f''(0) \tag{B8}$$

where the double dashes denote differentiation with respect to q. Performing the differentiations we get

$$\Delta M_2 = \sum_{N,t'}^{t}(t-t')(A_1(t') - A_2(t')) + \sum_{N,t'}^{t} N^2 l^2 (A_1(t') - A_2(t')) - \sum_{N,t'} t A_2(t') \tag{B9}$$

where

$$A_1(t') - A_2(t') = \sum_{s=1}^{t'} \Psi(s) \int_{-\pi}^{\pi} \mathrm{d}k (\cos k)^{(t'-s)} \exp(\mathrm{i}k(Nl - sl))(1 - \exp(\mathrm{i}kl)). \tag{B10}$$

Using the identity

$$\sum_{N=-\infty}^{\infty} \exp(\mathrm{i}kNl) = \sum_{N=-\infty}^{\infty} \delta\left(k - \frac{2\pi N}{l}\right) \tag{B11}$$

we can do the summation over N in the equation giving ΔM_2 and thus get

$$\begin{aligned} \Delta M_2 = &- \sum_{t'}^{t} \sum_{s=1}^{t'} \Psi(s) \int_{-\pi}^{\pi} \mathrm{d}k \sum_{N} \frac{\mathrm{d}^2}{\mathrm{d}k^2} (\cos k)^{t'-s} \\ &\times \exp(-\mathrm{i}ksl)(1 - \exp(\mathrm{i}kl))\delta(k - 2\pi N/l) \\ &- \sum_{t'}^{t} \sum_{s=1} t' \Psi(s) \sum_{N=-l/2}^{l/2} \cos\left(\frac{2\pi N}{l}\right)^{(t'-s)}. \end{aligned} \tag{B12}$$

We finally get for the real part of ΔM_2

$$\Delta M_2 = C_1 \sum_{t'}^{t} \sum_{s=1}^{t'} s\Psi(s) A(t'-s) - C_2 \sum_{t'}^{t} \sum_{s=1}^{t'} \Psi(s) A(t'-s) \tag{B13}$$

where

$$A(t'-s) = \sum_{N=-l/2}^{l/2} \cos\left(\frac{2\pi N}{l}\right)^{(t'-s)} \tag{B14}$$

and

$$\begin{aligned} C_1 &= 2l^2 \\ C_2 &= l^2 + 1. \end{aligned} \tag{B15}$$

The function $A(t'-s)$ is bounded by

$$| A(t'-s) | \leqslant l \tag{B16}$$

for all values of $t'-s$. The calculation of $A(t'-s)$ shows that it can take both positive and negative values, but they are distributed in such a way that ΔM_2 is always a positive quantity. Furthermore, for $t \to \infty$

$$\Delta M_2 \simeq C_1 \sum_{t'}^{t} \sum_{s=1}^{t'} s\Psi(s) - C_2 \sum_{t'}^{t} \sum_{s=1}^{t'} \Psi(s). \tag{B17}$$

The behaviour of the second moment can now be calculated by using the discrete relation (B13) without having to go to the continuous limit. It is also seen that the discrete model gives the same results as the continuous one as far as the results asymptotic in time are concerned. This can be easily seen by comparing the discrete relation (B17) with the relation for ΔM_2 given in appendix A, obtained for the continuous case. The results are even better for its Fourier–Laplace transform given in appendix C.

The correction to the normal random walk, due to the presence of the source term corresponding to traps is calculated in a similar way. The final result is

$$\Delta M_2 = -\alpha \sum_{t'}^{t} B(t'-1) + \alpha \sum_{t'}^{t} \sum_{s=1}^{t'} \frac{r(s)}{s} B(t'-s) \tag{B18}$$

where

$$B(t'-s) = \sum_{N=-l/2}^{l/2} \cos(2\pi N/l)^{t'-s} \cos(2\pi N n_T/l). \tag{B19}$$

The function $B(t'-s)$ has similar properties to the function $A(t'-s)$ defined above. It is such that the correction to the second moment due to the trap terms is always negative, thus giving rise to a decrease in the effective diffusion coefficient as expected. Furthermore for $t \to \infty$

$$\Delta M_2 \simeq -\alpha t + \alpha \sum_{t'}^{t} \sum_{s=1}^{t'} \frac{r(s)}{s}. \tag{B20}$$

It can be easily seen that this is just the discrete counterpart of the continuous relation for the case of traps, given in appendix C.

Appendix C.

In this appendix we give the complete solution of the continuous diffusion model given in section 2.2 in Fourier–Laplace space. Even though this solution is not easily transformed back into real space and used to give results for intermediate times, it can be illuminating as far as asymptotic results for the second moment of the probability distribution are concerned.

We start by taking into account only the accelerator modes term. If we take the Fourier transform of the diffusion equation proposed in section 2.2 we get

$$\begin{aligned} \frac{\partial}{\partial t}\hat{p}(k,t) + Dk^2\hat{p}(k,t) = \sum_N \mathrm{e}^{\mathrm{i}kNx_A} \sum_{s=1}^{[t]} \Psi(s)(p(Nx_A - sx_A, Ny_A, t-s) \\ -p(Nx_A - (s-1)x_A, Ny_A, t-s)) + \delta(t). \end{aligned} \tag{C1}$$

Manipulating the sum in the right-hand side of the above equation we get

$$\begin{aligned} \frac{\partial}{\partial t}\hat{p}(k,t) + Dk^2\hat{p}(k,t) = (1 - \exp(-\mathrm{i}k_x N x_A)) \sum_{s=1}^{[t]} \Psi(s) \exp(\mathrm{i}k_x s x_A) \\ \times \sum_N \exp(\mathrm{i}\boldsymbol{k}Nx_A) p(Nx_A, Ny_A, t-s) + \delta(t) \end{aligned} \tag{C2}$$

where $p(\hat{k}, t)$ is the Fourier transform of $p(x,t)$. Since the above model is formally two-dimensional, $\boldsymbol{k}$ is considered as a two-dimensional vector, and because the communication of the accelerator modes is done in the x direction only, it is the x-coordinate of $\boldsymbol{k}$ that enters the multiplicative factor in front of the Fourier transform of the source term.

Writing

$$p(Nx_A, t-s) = \int_{-\infty}^{\infty} \mathrm{d}q \exp(-\mathrm{i}Nx_Aq)\hat{p}(q, t-s) \tag{C3}$$

and using the fact that

$$\sum_N \exp(\mathrm{i}(k-q)x_AN) = \delta((k-q)x_A - 2\pi N) \tag{C4}$$

we can rewrite equation (C2) in the form

$$\frac{\partial}{\partial t}\hat{p}(k,t) + Dk^2\hat{p}(k,t) = (1-\exp(-\mathrm{i}k_xx_A))\sum_{s=1}^{[t]}\Psi(s)$$
$$\times \exp(\mathrm{i}k_xsx_A)\sum_N \hat{p}\left(k + \frac{2\pi}{x_A}N, t-s\right) + \delta(t). \tag{C5}$$

We now take the Laplace transform of this equation. This gives

$$u\hat{p}(k,u) + Dk^2\hat{p}(k,u) = a(1-\exp(-\mathrm{i}k_xx_A))\hat{\Psi}(u - \mathrm{i}k_xx_A)$$
$$\sum_N \hat{p}\left(k + \frac{2\pi}{x_A}N, u\right) + 1 \tag{C6}$$

where the convolution sum has been replaced by an integral. In the above equation, $\hat{\Psi}(u)$ is the Laplace transform of the function Ψ, and $\hat{p}(k,u)$ is the Fourier–Laplace transform of $p(x,y,t)$. The approximation of the convolution sum by an integral does not introduce new behaviour in the system, since the full dispersion relation of the discrete model using the discrete Fourier transform and the z-transform, where one makes no approximations of this sort, is analogous to that obtained here for the continuous model and gives similar asymptotic results. The derivation of the dispersion relation for the discrete model is similar to the one presented here, only it does not involve any of the approximations necessary to be introduced in the continuous case.

We solve the operator equation for $\hat{p}(k,u)$ using the iteration scheme

$$\hat{p}^{(m)}(k,u) = G^0(k,u) + \alpha f(k)G^0(k,u)\widehat{\Psi}(u - \mathrm{i}k_xx_A) \quad \times \sum_N \hat{p}^{(m-1)}\left(k + \frac{2\pi}{x_A}N, u\right) \tag{C7}$$

where

$$G^0(k,u) = \frac{1}{u + Dk^2} \tag{C8}$$

and

$$f(k) = 1 - \exp(-\mathrm{i}k_xx_A) \tag{C9}$$

As the zeroth-order approximation, we use $\hat{p}^0(k,u) = G^0(k,u)$ which is the Fourier–Laplace transform of the diffusion equation in the case of no sources ($\alpha = 0$).

It is clear that this iterative scheme is just the Fourier–Laplace space version of our perturbative solution of the diffusion equation employed in section 3. The advantage of using this method in the Fourier–Laplace space for the solution of the dispersion relation is that we can get iterations of this scheme up to an arbitrary order, thus getting a formal series in powers of α for the complete solution of the problem. The full solution to the problem is then

$$\hat{p}(k,u) = G^0(k,u) + \sum_{n=1}^{\infty}\alpha^n\hat{p}_n(k,u) \tag{C10}$$

where

$$\hat{p}_n(k,u) = f(k)G^0(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A) \sum_{m_1 \ldots m_n} \Pi_{s=1}^{n-1} f\left(k + A\sum_{i=1}^{s} m_i\right)$$
$$G^0(k + A\sum_{i=1}^{s} m_i, u)\hat{\Psi}\left(u - \mathrm{i}k_x x_A - \mathrm{i}A\sum_{i=1}^{s} m_i\right) G^0\left(k + A\sum_{i-1}^{n} m_i, u\right) \tag{C11}$$

and $A = 2\pi/x_A$.

It is easy to see that the full solution to the problem gives

$$\hat{p}(0,u) = \frac{1}{u} \tag{C12}$$

which is equivalent to the conservation of particles.

We now use equation (C10) to get the Laplace transform for the second moment of the probability distribution. As is well known, second moments are given by

$$M_2(u) = -\frac{\mathrm{d}^2\hat{p}(k,u)}{\mathrm{d}k^2}|_{k=0}\,. \tag{C13}$$

Differentiating $\hat{p}(k,u)$ twice we get

$$\frac{\mathrm{d}^2\hat{p}(k,u)}{\mathrm{d}k^2} = -\frac{2D}{(u+Dk^2)^2} + \frac{2D^2k^2}{(u+Dk^2)^3} + \sum_{n=1}^{\infty} \alpha^n \frac{\mathrm{d}^2\hat{p}_n(k,u)}{\mathrm{d}k^2} \tag{C14}$$

where

$$\begin{aligned}\frac{\mathrm{d}^2\hat{p}_n(k,u)}{\mathrm{d}k^2} = {} & f''(k)G^0(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A, u)F_1(k,u) \\ & + f'(k)G^{0'}(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A, u)F_1(k,u) \\ & + f'(k)G^0(k,u)\hat{\Psi}'(u - \mathrm{i}k_x x_A, u)F_1(k,u) \\ & + f'(k)G^0(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A, u)G_1(k,u) \\ & + f'(k)G^0(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A, u)G_2(k,u) \\ & + f'(k)G^0(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A, u)G_3(k,u) \\ & + f'(k)G^0(k,u)\hat{\Psi}(u - \mathrm{i}k_x x_A, u)G_4(k,u)\end{aligned} \tag{C15}$$

and

$$\begin{aligned}F_1(k,u) = {} & \sum_{m_1 \ldots m_n} \prod_{s=1}^{n-1} f\left(k + A\sum_{i=1}^{s} m_i\right) G^0\left(k + A\sum_{i=1}^{s} m_i, u\right) \\ & \times \hat{\Psi}\left(u - \mathrm{i}x_A k_x - \mathrm{i}A\sum_{i=1}^{s} m_i\right) G^0\left(k + A\sum_{i=1}^{n} m_i, u\right)\end{aligned} \tag{C16}$$

$$\begin{aligned}G_1(k,u) = {} & \sum_{s_0=1}^{n-1} \sum_{m_1 \ldots m_n} f'\left(k + A\sum_{i=1}^{s_0} m_i\right) G^0\left(k + A\sum_{i=1}^{s_0} m_i\right) \\ & \times \hat{\Psi}\left(u - \mathrm{i}x_A k_x - \mathrm{i}A\sum_{i=1}^{s_0} m_i\right) \\ & \prod_{s=1, s\neq s_0}^{n-1} f\left(k + A\sum_{i=1}^{s} m_i\right) G^0\left(k + A\sum_{i=1}^{s} m_i, u\right) \\ & \times \hat{\Psi}\left(u - \mathrm{i}x_A k_x - \mathrm{i}A\sum_{i=1}^{s} m_i\right) G^0\left(k + A\sum_{i=1}^{n} m_i, u\right)\end{aligned} \tag{C17}$$

$G_2(k,u)$ is the same as the above but with $G^{0'}$ instead of f', $G_3(k,u)$ is again the same as the above but with $\hat{\Psi}'$ instead of f' and finally $G_4(k,u)$ is the same as F_t but the last G^0 function is differentiated with respect to k, that is, it is substituted by a $G^{0'}$. It is obvious that the terms containing f' and $\hat{\Psi}'$ are non-zero only if derivatives with respect to k_x are taken.

The asymptotic behaviour of the second moment is given in the limit $u \to 0$ and $k = 0$. The terms diverging as $u \to 0$ are those of interest. Terms of the form $G^0(k + A\sum_{s=1}^{q} m_s, u)$ are going to diverge as $u \to 0$, only if $\sum_{s=1}^{q} m_s = 0$. However, because of the presence of terms of the form $f(\sum_{s=1}^{q} m_s)$ in the series giving the Laplace transform of the second moment as $k = 0$, and of the property $f(0) = 0$, we are not free to have as many G^0s diverging at $u \to 0$ as we like. Observing the structure of the series and taking into account that $f'(0) \neq 0$ and $G^{0'}(0) = 0$, we see that the only possible diverging terms as $u \to 0$ are such that

$$M_{2x}(u) = \frac{1}{u^2} + C_1\frac{\hat{\Psi}'(u)}{u^2} + C_2\frac{\hat{\Psi}^2(u)}{u^3} \tag{C18}$$

whereas $M_{2y}(u)$ is just equal to $1/u^2$. The first term in the above sum is simply the normal diffusive behaviour $M_2 \equiv t$. The second term corresponds in real space to a behaviour of the form

$$\Delta M_2(t) \equiv \int_0^t \int_0^\tau s\Psi(s)\mathrm{d}s\,\mathrm{d}\tau \tag{C19}$$

and the third term to a behaviour of the form

$$\Delta M_2(t) \equiv \int_0^t (t-u)^2 \Psi \star \Psi(u)\mathrm{d}u \tag{C20}$$

where $\star$ denotes the convolution product.

Thus the full solution to the diffusion model for the second moment in x is given by equations (C18)–(C20).

In the case where the trap term is introduced into the system, the same procedure should be followed. By taking the Fourier–Laplace transforms of the continuous equations we get

$$u\hat{p}(k,u) + Dk^2\hat{p}(k,u) = \frac{\alpha}{2}k^2(1-R(u))\sum_N \mathrm{e}^{inx_0}\hat{p}(k+N,u) + 1 \tag{C21}$$

where $R(u)$ is the Laplace transform of $r(s)/s$ and we have assumed that the traps are situated on a periodic lattice which without loss of generality can be taken as $x_0 + 2\pi N$.

This is an equation for $\hat{p}(k,u)$ which can be solved using the following iterative scheme

$$\hat{p}^{(m+1)}(k,u) = G^0(k,u) + \frac{\alpha}{2}k^2(1-R(u))G^0(k,u)\sum_N \mathrm{e}^{iNx_0}\hat{p}^{(m)}(k+N,u) \tag{C22}$$

where

$$G^0(k,u) = \frac{1}{u + Dk^2} \tag{C23}$$

is the Fourier–Laplace transform of the Green function for the diffusion process when $\alpha = 0$. In the above, k may be considered as a vector or a scalar according to the dimension of the diffusion process. Since the trapping process does not create a prefered direction, as in the case of the accelerator modes where the streaming was defining a prefered direction, it is not of great importance to think of k as a vector.

Starting with $\hat{p}^{(0)}(k,u) = G^0(k,u)$, we get the full solution

$$\hat{p}(k,u) = G^0(k,u) + \frac{1}{2}\sum_{s=1}^{\infty}\alpha^s k^2(1-R(u))^s G^0(k,u)F_s(k,u) \tag{C24}$$

where

$$F_s(k,u) = \sum_{n_1 \ldots n_s} \exp(\mathrm{i}(n_1 + \ldots + n_s)x_0)(k+n_s)^2 G^0(k+n_s,u)\ldots$$
$$\times(k+n_s+n_{s-1}+\ldots+n_2)^2 G^0(k+n_s+n_{s-1}+\ldots+n_2,u)$$
$$\times G^0(k+n_s+\ldots+n_1,u) \tag{C25}$$

The second moment we are interested in, is equal to $-\hat{p}''(0,u)$ which is

$$\hat{p}''(0,u) = G^{0''}(0,u) + \sum_{s=1}^{\infty} \alpha^s (1-R(u))^s G^0(0,u) F_s(0,u). \tag{C26}$$

We are interested in terms diverging as $u \to 0$, because these are the terms which give asymptotic contributions in time. It can be seen that the only case where $F_s(0,u)$ can diverge is when $n_1 + \ldots + n_s = 0$ while all the other sums $n_m + \ldots + n_s \neq 0$ where $m \geqslant 2$. This gives a divergence of $1/u$ which is due to the $G^0(0,u)$ term.

So, $\hat{p}''(0,u)$ diverges as

$$-\frac{1}{u^2} + \sum_{s=1}^{\infty} \alpha^s \frac{(1-R(u))^s}{u^2} \tag{C27}$$

The correction of the second moment due to the trap terms is then

$$\Delta M_2 = -\sum_{s=1}^{\infty} \alpha^s \frac{(1-R(u))^s}{u^2}. \tag{C28}$$

This gives a contribution

$$\Delta M_2 = -\frac{\alpha}{1-\alpha}\frac{1}{u^2} + \sum_{s=1}^{\infty} A_s \frac{R(u)^s}{u^2} \tag{C29}$$

where A_s are constant terms that can be obtained from the expansion of $(1-R(u))^s$.

Transforming back to time, this relation becomes

$$\Delta M_2(t) = -\frac{\alpha}{1-\alpha} t + \sum_{s=1}^{\infty} A_s \int^t \int^{t'} \left(\frac{r(\tau)}{\tau}\right)^{*s} \mathrm{d}\tau\, \mathrm{d}t' \tag{C30}$$

where f^{*s} denotes the convolution of f, s-times with itself.

Appendix D.

In this appendix, the asymptotic results for M_2x are obtained for various forms of the waiting time probability distribution $\psi(s)$.

1. *Power law.* Assume the the trapping probability distribution in the accelerator modes behaves asymptotically with time as a power law

$$\psi(t) \sim t^{-1-\beta} \qquad t \to \infty \qquad 1 < \beta < 2. \tag{D1}$$

Then, from the definition of $\Psi(t)$ we see that

$$\Psi(t) \sim t^{-\beta} \qquad t \to \infty. \tag{D2}$$

In the previous appendix it was shown that in the presence of accelerator modes, the second moment has the following corrections as t$\to \infty$

$$\Delta M_2^{(1)}(t) \sim \int_{\tau_c}^{t} \int_{\tau_c}^{\tau} s\Psi(s)\mathrm{d}s\, \mathrm{d}\tau \tag{D3}$$

and

$$\Delta M_2^{(2)}(t) \sim \int_{t_c}^{t} (t-u)^2 \Psi \star \Psi(u)\, du \tag{D4}$$

where τ_c and t_c are times for which our asymptotic forms for $\Psi(t)$ are valid. For $\Psi(t) \sim t^{-\beta}$ as $t \to \infty$ it is easy to see that

$$M_2^{(1)}(t) \sim t^{3-\beta}. \tag{D5}$$

The convolution $\Psi \bullet \Psi$ will behave asymptotically as $t^{1-2\beta}$ for $t \to \infty$ so that

$$\Delta M_2^{(2)}(t) \sim t^{4-2\beta}. \tag{D6}$$

For $1 < \beta < 2$ the dominant contribution as $t \to \infty$ is that of $M_2^{(1)}$.

In the case where trap terms are introduced the asymptotic behaviour for the corrections to the second moment is

$$\Delta M_2^{(1)}(t) \simeq -\frac{\alpha}{1-\alpha} t \tag{D7}$$

and

$$M_2(t) \simeq \int_{t_c}^{t} \int_{\tau_c}^{t'} \left(\frac{r(\tau)}{\tau}\right)^{\star s} d\tau\, dt' \tag{D8}$$

which for $r(t) \sim t^{-1-\beta}$ as $t \to \infty$ behaves as

$$M_2(t) \sim t^{1-s\beta} \tag{D9}$$

which for every $s \geqslant 1$ decay to 0 as $t \to \infty$.

2.*Exponential form*. If the distribution function $\bar{\psi}(t)$ decays exponentially then in general $\psi(t)$ will decay as $\exp(-\lambda t)$ as $t \to \infty$.

In that case

$$\Delta M_2^{(1)} \sim \int^{t} \int^{s} s \exp(-\lambda s) ds\, dt \sim \exp(-\lambda t) \tag{D10}$$

So the term $\Delta M_2^{(1)}$ will not contribute to $t \to \infty$ in the case of an exponential trapping distribution. The same happens with the term $\Delta M_2^{(2)}$. If $\psi(t) \sim t^{-n} \exp(-\lambda t)$ then $\Psi(t) < f(t) = \exp(-\lambda t)$ for $t > 1$ and $\Psi \star \Psi(t) < f \star f(t) = t \exp(-\lambda t)$. Then

$$\Delta M_2^{(2)} < \int^{t} (t-u)^2 u \exp(-\lambda u) du \tag{D11}$$

and this last integral decays exponentially to zero as $t \to \infty$. So $\Delta M_2^{(2)}$ again will not contribute to the second moment for $t \to \infty$ in the case of an exponential trapping distribution.

The same is true for the correction term due to the presence of traps.

References

[1] Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (New York: Springer)
[2] Chirikov B V 1979 *Phys. Rep.* **52** 263
[3] Karney C F F, Rechester A B and White R B 1982 *Physica* **4D** 425
[4] Zaslavskii G M, Sagdeev R Z, Usikov D A and Chernikov A A 1989 *Sov. Phys. Usp.* **31** 887 and references therein
[5] Yannacopoulos A N and Rowlands G 1991 *Phys. Lett.* **155A** 133
[6] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 1 (New York: McGraw-Hill)
[7] Ishizaki R, Horita T, Kobayashi T and Mori H 1991 *Prog. Theor. Phys.* **85** 1013
[8] Meiss J D and Ott E 1986 *Physica* **20D** 387
[9] Zaslavsky G M and Tippett M K 1991 *Phys. Rev. Lett.* **67** 3251